

# A Network Topology Dependent Upper Bound on the Number of Equilibria of the Kuramoto Model

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**Abstract**—We begin with formulating the stationary equations of the Kuramoto model as a system of polynomial equations in a novel way. Then, based on an algebraic geometric root count, we give a prescription of computing an upper bound on the number of equilibria of the Kuramoto model for the most general case, i.e., defined on an arbitrary graph and having generic values of natural frequencies and inhomogeneous couplings. We demonstrate with computational experiments utilizing the numerical polynomial homotopy continuation method that our bound is tight for the number of complex equilibria for the Kuramoto model in the most general case. We then discuss the implications of our results in relation to finding all the real equilibria of the Kuramoto model.

## I. INTRODUCTION

The Kuramoto model [1]–[4] is a prototypical model for studying many phenomena including the synchronization power systems, neural networks, chemical oscillators, particle coordinations, rhythmic applause, and so on. The general Kuramoto model is given by the differential equation:

$$\frac{d\theta_i}{dt} = \omega_i - \frac{1}{N} \sum_{j=1}^N K_{i,j} \sin(\theta_i - \theta_j), \text{ for } i = 1, \dots, N, \quad (1)$$

where  $N$  is the number of oscillators,  $K_{i,j}$  is the coupling strength between the  $i$ -th and  $j$ -th oscillators. The matrix  $K = [K_{i,j}]$  may also be viewed as the adjacency matrix for the underlying weighted graph.  $\Omega = (\omega_1, \dots, \omega_N)$  contains the natural frequencies of the  $N$  oscillators. Of crucial importance in studying the phase space of this system of ordinary differential equations are the equilibria which are values of  $\theta_1, \dots, \theta_N$  for which  $\frac{d\theta_i}{dt} = 0$  for all  $i = 1, \dots, N$ . The stability analysis of the equilibria can reveal the behavior of the dynamical system near the equilibria. One can also study synchronization phenomenon of the Kuramoto model with the knowledge of the equilibria of the model [5]. Interpreting as the special case [6] of the power flow equations the equilibria of the Kuramoto model provide

crucial information for planning and designing power grids.

Since the equilibrium conditions are invariant under transformations of the form  $\theta_i \rightarrow \theta_i + \alpha$  for all  $\theta_i$  with any fixed  $\alpha \in (-\pi, \pi]$ , they possess infinitely many solutions. To remove this degree of freedom, we fix  $\theta_N = 0$  and remove the  $N$ -th equation  $\frac{d\theta_N}{dt} = 0$ . Thus, we are left with  $N - 1$  nonlinear equations in  $N - 1$  angles.

The problem of counting the number of equilibria of the finite  $N$  Kuramoto model has been tackled for different graphs such as complete graphs [7], nearest-neighbor coupling on one-dimensional lattice graphs [7]–[11] and with long-range interactions [12], two-dimensional lattice graphs [8], [13]–[17], and three-dimensional lattice graphs [17]–[19]; and the homogeneous frequencies case (also known as the power flow equations for the lossless power system) [20], in different disguises.

For  $N = 3$ , the Kuramoto model on a complete graph of 3 nodes was shown to have at most 6 equilibria [21]–[23]. Several necessary and sufficient conditions for equilibria to exist and other conjectures on the types of equilibria for different graphs are provided in [24]–[32] though some of which were disproved by counterexamples in [5].

It was also shown in Ref. [7] that even after fixing the trivial zero mode by setting  $\theta_N = 0$ , there existed infinitely many equilibria (also called *incoherent manifolds* [33]). Surprisingly, such infinite families of equilibria were also shown to exist in the one-, two- and three-dimensional lattice model with nearest neighbour interaction in [8], [15], [34], where it was demonstrated that the number of infinite families of equilibria exponentially grows with  $N$ .

As a significant leap forward from the recent studies of the Kuramoto model and the closely related load flow equations (for power networks) from algebraic view points by the authors [5], [35], in the present contribution, we establish an upper bound on the number of equilibria for (1) using a novel polynomial formulation of the and the theory of Bernshtein-Kushnirenko-Khovanskii (BKK) bound [36]–[38]. The advantage of this bound over existing bounds is that it takes into consideration of the sparsity of the connections in the underlying network. We present strong numerical evidence that for many Kuramoto models, the proposed bound is best possible bound for the number of complex equilibria. In § II-A, we describe a novel polynomial formulation of the equilibrium equations of the Kuramoto model. After

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reviewing existing bounds on the number of equilibria in § II-B, we propose an upper bound on the number of equilibria based on the theory of BKK bound in § II-C. We compute this bound for the Kuramoto models defined on different graphs. Employing the numerical polynomial homotopy continuation method, we provide a strong numerical evidence to demonstrate that this bound to be tight in these cases. In IV, we discuss the implications of our results and conclude.

## II. UPPER BOUND ON THE NUMBER OF EQUILIBRIA

In this section we discuss a novel polynomial formulation of the Kuramoto model and provide an upper bound based on the theory of BKK bound.

### A. Polynomial Formulations

In studying nonlinear systems of equations like the equilibrium equations of (1), it is a common practice to first transform them into algebraic equations which would allow the use of powerful tools from algebraic geometry. Previously (e.g., [5], [8], [16], [20], [39]), the equilibrium conditions of Eqs. (1) were transformed into a system of polynomial equations by first using the identities  $\sin(\theta_i - \theta_j) = \sin \theta_i \cos \theta_j - \sin \theta_j \cos \theta_i$  and then the substitution  $s_i := \sin \theta_i$  and  $c_i := \cos \theta_i$  for all  $i = 1, \dots, N-1$ , as well as adding the constraint equations  $s_i^2 + c_i^2 - 1 = 0$ , for all  $i = 1, \dots, N-1$ , to finally obtain the following system of polynomial equations:

$$\begin{aligned} \omega_i - \frac{1}{N} \sum_{j=1}^N K_{i,j} (s_i c_j - s_j c_i) &= 0 \\ s_i^2 + c_i^2 - 1 &= 0, \end{aligned} \quad (2)$$

for  $i = 1, \dots, N-1$ .

In the present work, we adopt a different transformation: using the trigonometric identity,  $\sin(\theta_i - \theta_j) = \frac{1}{2\mathbb{I}}(e^{\mathbb{I}(\theta_i - \theta_j)} - e^{-\mathbb{I}(\theta_i - \theta_j)})$  where  $\mathbb{I} := \sqrt{-1}$  is the imaginary unit, the equilibrium equations of (1) becomes

$$\omega_i - \sum_{j=1}^N \frac{K_{i,j}}{\mathbb{I}N} (e^{\mathbb{I}\theta_i} e^{-\mathbb{I}\theta_j} - e^{-\mathbb{I}\theta_i} e^{\mathbb{I}\theta_j}) = 0. \quad (3)$$

To formulate the above system of equations as an algebraic system, we let

$$x_i := e^{\mathbb{I}\theta_i} \quad \text{and} \quad y_i := e^{-\mathbb{I}\theta_i}, \quad (4)$$

for all  $i = 1, \dots, N-1$ . With this substitution (3) becomes an enlarged system of  $2(N-1)$  equations in  $2(N-1)$  variables

$$\begin{aligned} \sum_{j=1}^N \frac{K_{i,j}}{\mathbb{I}N} (x_i y_j - x_j y_i) &= \omega_i \quad \text{for } i = 1, \dots, N-1 \\ x_i y_i &= 1 \quad \text{for } i = 1, \dots, N-1. \end{aligned} \quad (5)$$

For example, with  $N = 3$ , the system becomes

$$\begin{aligned} \frac{K_{1,2}}{3\mathbb{I}}(x_1 y_2 - x_2 y_1) + \frac{K_{1,3}}{3\mathbb{I}}(x_1 - y_1) - \omega_1 &= 0 \\ \frac{K_{2,1}}{3\mathbb{I}}(x_2 y_1 - x_1 y_2) + \frac{K_{2,3}}{3\mathbb{I}}(x_2 - y_2) - \omega_2 &= 0 \\ x_1 y_1 - 1 &= 0 \\ x_2 y_2 - 1 &= 0. \end{aligned} \quad (6)$$

It can be readily verified that the equilibria of (1) (with the translation symmetry removed) are in one-to-one correspondence with the special solutions of the above system (5) that satisfy the additional restriction that  $|x_i| = |y_i| = 1$  for  $i = 1, \dots, N-1$ .

### B. Existing Bounds

Via the transformations given in (2) or (5), the problem of counting equilibria of the Kuramoto model is turned into a *root counting* problem for systems of polynomial equations. A well known bound on the root count of a system of polynomial equations is the *Bézout bound* which is simply the product of the degrees of all the equations. In the example shown in (6), since each of the four equations is quadratic (degree 2), the highest possible number of isolated solutions as given by the Bézout bound is therefore  $2^4 = 16$ . In general, the Bézout bound for the system (5) is  $2^{2(N-1)}$ .

The Bézout bound is a basic result in *intersection theory* [40]. Based on this theory, a more refined root count can be derived for the equilibrium equations for (1). This was first studied in Ref. [20] in which it was shown to be the binomial coefficient  $\binom{2(N-1)}{N-1}$ , which will be referred to as the *binomial bound*.<sup>1</sup>

### C. BKK Bound

In this subsection, we briefly review the Bernstein–Khovanskii–Kushnirenko (BKK) bound [36]–[38] for the number of isolated nonzero complex solutions which is a refinement of the Bézout bound that takes into consideration the monomials that appear in the polynomial system: Given a polynomial, each of its terms give rise to an *exponent vector*. For instance, for the term  $x^3 y^2 z^1$ , the exponent vector is simply the vector whose entries are the exponents of  $x, y$  and  $z$ , respectively, i.e.,  $(3, 2, 1)$ . The choice of this ordering is inconsequential as long as it is kept the same for each equation. The set of all exponent vectors derived from the nonzero terms of an polynomial equation is called the *support* of that equation. For example, if we arrange the variables in the order of  $(x_1, y_1, x_2, y_2)$ , then the supports of the four equations in (6) are

$$\begin{aligned} S_1 &= \{(1, 0, 0, 1), (0, 1, 1, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 0, 0)\} \\ S_2 &= \{(0, 1, 1, 0), (1, 0, 0, 1), (0, 0, 1, 0), (0, 0, 0, 1), (0, 0, 0, 0)\} \\ S_3 &= \{(1, 1, 0, 0), (0, 0, 0, 0)\} \\ S_4 &= \{(0, 0, 1, 1), (0, 0, 0, 0)\}. \end{aligned}$$

<sup>1</sup>In Ref. [20], the system (1) was termed as a special case of the power flow equations. Later on, the same binomial bound was proposed for the complete power flow equations [41] (see [42] for an alternative proof for this bound).

A *convex set* is a set of points in which the line segment connecting any pair of points in the set also lie in that set. The *convex hull* of a set is the minimal convex set containing that set. For a polynomial, the convex hull of its support is known as the *Newton polytope* of that polynomial. In the study of convex polytopes, the *mixed volume* of several polytopes is an important concept. In the simplest case, the mixed volume of two line segments on the plane is precisely the area of the parallelogram spanned by translations of these two line segments. The general definition is included in the Appendix. Bernstein’s Theorem [36] provides an upper bound for the number solutions to a system of polynomial equations in terms of mixed volume:

*Theorem 1 (Bernshtein [36]):* Given a system of  $n$  polynomial equations in  $n$  variables, the number of isolated complex solutions for which no variable is zero is bounded above by the mixed volume of the Newton polytopes of the equations.

This upper bound is known as the BKK bound, after a circle of closely related works by Bernstein[36], Kushnirenko[37], and Khovanskii[38]. Recall that via the change of variables given in (4), each (real) equilibrium of (1) corresponds to a unique nonzero complex solution of (5). Therefore the BKK bound given above provides an upper bound to the number of isolated (real) equilibria.

A crucial fact is that the BKK bound is *generically exact*: when the coefficients in the polynomial system are chosen at random, *with probability one* the number of isolated complex solutions for which no variable is zero is exactly the BKK bound. In the polynomial formulation of the Kuramoto model given in (5), if certain relations are imposed on the coefficients (e.g., the coefficients of  $x_1y_2$  and  $x_2y_1$  in (6) must be the same) the generic exactness still holds true with a mild additional condition:

*Theorem 2:* If there exists a choice of  $K_{i,j}$ ’s and  $\omega_i$ ’s for which the number of nonzero complex solutions of (5) is the BKK bound, then for almost all choices of complex  $K_{i,j}$ ’s and  $\omega_i$ , the number of nonzero complex solutions of (5) will be the BKK bound.

In other words, among the systems (5) for all possible choices of the  $K_{i,j}$ ’s and  $\omega_i$ ’s, if the BKK bound is attainable then it must also be generically exact. In the following, we shall compute the BKK bound for the polynomial system (5) induced by a number of graphs. Then, the attainability and hence the generic exactness of the BKK bound in each case is verified by solving the system (5) for some specific chosen set of  $K_{i,j}$ ’s and  $\omega_i$ ’s.

### III. RESULTS

In this section, we provide results for the Kuramoto model on different graphs. We compare the BKK bound for both formulations (2) and (5) with the help of the computational packages *HOM4PS-3.0* [43], [44] and *Bertini* [45], [46]. We also compare the bounds with the corresponding binomial bound.

To analyze how tight the above mentioned bounds are we provide results from our numerical experiments using the NPHC method. The NPHC method guarantees that one will obtain all isolated complex solutions for a system of polynomial equations by following the strategy as briefly explained below[47], [48]: to solve a system of polynomial equations, one starts with an upper bound on the number of complex solutions of the system. Then, another system is created such that the system has exactly the same number of complex solutions as the upper bound, and it is easy to solve. Finally, each solution of this new system is evolved over a single parameter towards the system to be solved. The so-called gamma-trick is employed to ensure to achieve all the isolated complex solutions of the system.

#### A. BKK for the Complete Graphs

The complete graph is a graph in which each node is connected to all other nodes. Hence, this is the most densely connected graph. We compare the Bézout bound, binomial bound, the BKK bound for the traditional polynomial formulation (2), the BKK bound for the new formulation (5), and the actual number of complex solutions (with 100 random samples for each graph) in Table I.

#### B. BKK for the Path Graphs

Now we turn our attention to one of the most sparsely connected graphs — path graph. In a path graph, the  $i$ -th node for  $1 < i < N - 1$  is connected to two of its neighbors: the  $(i - 1)$ -th node and the  $(i + 1)$ -th node forming a path. The results are given in Table II.

#### C. BKK for the Ring Graphs

A ring graph is a graph in which each node is connected only to its nearest neighbors, same as the path graph, and the last node is connected to the first node making a closed loop. The results are in Table III.

#### D. BKK for Erdős-Rényi Random Graphs

The Erdős-Rényi model for random graphs was initially studied in [49]–[51]. There are two formulations of the model, but we will use the convention given originally by Gilbert, that is the existence of each edge is i.i.d. with probability  $p$ . Such graphs are denoted as  $G(N, p)$  where  $N$  is the number of vertices of the graph and  $p$  is the probability that any edge exists.

We construct a random graph  $G(N, p)$  by initially creating an  $N \times N$  empty adjacency matrix,  $A$ . Then, for each  $A_{i,j}$  where  $i > j$ , we test for the existence of edge  $E_{i,j}$  with the given probability  $p$  and update  $A_{i,j}$  and  $A_{j,i}$  accordingly.

Connectedness of a graph is necessary to have non-trivial equilibria for the Kuramoto model. Since the probability that a random graph  $G(N, p)$  is connected increases if  $p > p_c = \ln(N)/N$ , we let  $p$  vary between  $p_c$  and 1. Testing for connectedness is done by computing

No. of Nodes	3	4	5	6	7	8	9	10	11	12	13	14	15
Bézout	16	64	256	1024	4096	16384	65536	262144	1048576	4194304	16777216	67108864	268435456
Binomial	6	20	70	252	924	3432	12870	48620	184756	705432	2704156	10400600	40116600
BKK for (2)	6	40	192	864	3712	15488	63488	257536	1038336	4171776	16728064	67002368	268206080
BKK for (5)	6	20	70	252	924	3432	12870	48620	184756	705432	2704156	10400600	40116600
Generic root count	6	20	70	252	924	3432	12870	48620	184756	705432	2704156	10400600	40116600

TABLE I

COMPARISON AMONG DIFFERENT BOUNDS ON THE NUMBER OF EQUILIBRIA AND THE ACTUAL NUMBER OF COMPLEX SOLUTIONS FOR  
GENERIC PARAMETER-VALUES FOR THE KURAMOTO MODEL ON THE COMPLETE GRAPH.

Nodes	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Bézout	16	64	256	1024	4096	16384	65536	262144	1048576	4194304	16777216	67108864	268435456	1073741824
Binomial	6	20	70	252	924	3432	12870	48620	184756	705432	2704156	10400600	40116600	155117520
BKK for (2)	8	24	80	256	832	2688	8704	28160	91136	294912	954368	3088384	9994240	32342016
BKK for (5)	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768
Generic root count	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768

TABLE II

COMPARISON AMONG DIFFERENT BOUNDS ON THE NUMBER OF EQUILIBRIA AND THE ACTUAL NUMBER OF COMPLEX SOLUTIONS FOR  
GENERIC PARAMETER-VALUES FOR THE KURAMOTO MODEL ON THE PATH GRAPH.

Nodes	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Bézout	16	64	256	1024	4096	16384	65536	262144	1048576	4194304	16777216	67108864	268435456	1073741824
Binomial	6	20	70	252	924	3432	12870	48620	184756	705432	2704156	10400600	40116600	155117520
BKK for (2)	8	24	80	256	832	2688	8704	28160	91136	294912	954368	3088384	9994240	32342016
BKK for (5)	6	12	30	60	140	280	630	1260	2772	5544	12012	24024	51480	102960
Generic root count	6	12	30	60	140	280	630	1260	2772	5544	12012	24024	51480	102960

TABLE III

COMPARISON AMONG DIFFERENT BOUNDS ON THE NUMBER OF EQUILIBRIA AND THE ACTUAL NUMBER OF COMPLEX SOLUTIONS FOR  
GENERIC PARAMETER-VALUES FOR THE KURAMOTO MODEL ON THE RING GRAPH.

the size of the largest component of  $G(N, p)$  and testing for equality with  $N$ .

For both formulations of the Kuramoto model, the Bézout bound is always  $2^{2(N-1)}$ . For all the random Erdős-Rényi connected graphs with  $N = 5$  and  $6$  that we tested, we found that the complex solution count of the associated polynomial system under the Kuramoto model using the formulation of (5) was the same as the BKK bound. For  $N = 5$  and  $6$ , we calculated the number of complex roots for 10 random instances of the Kuramoto model using (2) and (5). Tables IV and V compare the BKK root count of polynomial systems arising from the Kuramoto model associated to 10 instances of random Erdős-Rényi connected graphs using the two formulations.

#### IV. DISCUSSION AND CONCLUSION

In this article, we have translated the stationary equations of the Kuramoto model to the polynomial form in a different way than the traditional polynomial formulation to utilize complex algebraic geometry results. We have considered a general form of the Kuramoto model, i. e. , with arbitrary natural frequencies and on graphs with arbitrary weights. We then provide a prescription to compute a tight upper bound, called the BKK bound, on the number of equilibria of the model for a given graph topology, i.e., the bound captures the graph topology accurately.

We first reproduced the previously available upper bound,  $\binom{2(N-1)}{N-1}$ , by computing the BKK bound for the complete graph with arbitrary (and inhomogeneous)

coupling strengths and natural frequencies: the binomial bound is the tightest possible bound for the Kuramoto model on the complete graph. The previous upper bound, however, does not take into account the sparsity of the graphs unlike ours. For other sparser graphs such as path graphs, ring graphs and Erdős-Rényi random graphs, the BKK bound for the new polynomial formulation is significantly lower than the binomial bound as well as the BKK bound for the traditional polynomial formulation.

Furthermore, with extensive numerical experiments with the NPHC method, which guarantees to find all isolated complex solutions of systems of polynomial equations, we find that for generic natural frequencies and weights the number of complex solutions is always equal to the BKK bound of the new polynomial formulation. Hence, this is the best possible upper bound on the number of complex solutions for generic values of natural frequencies and inhomogeneous couplings.

Our prescription is also constructive in solving the stationary equations of the model in that we not only provide the tight-most upper bound but also the optimum way of constructing a starting system for the NPHC method. Because our numerical experiments suggest that the BKK bound is always equal to the number of complex solutions for generic parameter values, no path diverges in the NPHC computation.

This tight upper bound not only helps the NPHC method to solve the corresponding systems more efficiently, but it also provides a concrete stopping criterion to other stochastic nonlinear equation solving method. Our results also give rise to the following interesting

Sample no.	1	2	3	4	5	6	7	8	9	10
Bézout bound	256	256	256	256	256	256	256	256	256	256
Binomial	70	70	70	70	70	70	70	70	70	70
BKK for (2)	48	144	112	80	16	112	80	48	80	48
BKK for (5)	24	32	24	24	16	24	16	24	32	16
Complex Count	24	32	24	24	16	24	16	24	32	16

TABLE IV

COMPARISON AMONG DIFFERENT BOUNDS ON THE NUMBER OF EQUILIBRIA AND THE ACTUAL NUMBER OF COMPLEX SOLUTIONS FOR GENERIC PARAMETER-VALUES FOR THE KURAMOTO MODEL ON THE RANDOM GRAPH OF 5 NODES FOR 10 SAMPLES.

Sample no.	1	2	3	4	5	6	7	8	9	10
Bézout bound	1024	1024	1024	1024	1024	1024	1024	1024	1024	1024
Binomial	252	252	252	252	252	252	252	252	252	252
BKK for (2)	128	320	288	544	480	544	224	160	160	608
BKK for (5)	48	48	48	88	104	144	48	64	32	140
Complex Count	48	48	48	88	104	144	48	64	32	140

TABLE V

COMPARISON AMONG DIFFERENT BOUNDS ON THE NUMBER OF EQUILIBRIA AND THE ACTUAL NUMBER OF COMPLEX SOLUTIONS FOR GENERIC PARAMETER-VALUES FOR THE KURAMOTO MODEL ON THE RANDOM GRAPH OF 6 NODES FOR 10 SAMPLES.

and mathematically challenging questions: can there be some values of  $K_{i,j}$ 's and  $\omega_i$ 's so that the number of real equilibria attains the BKK bound for the given graph? If not, what is the maximum number of real solutions a particular graph can have? Though there are a few algebraic geometry based methods[52] have been developed to answer such questions, they can not yet deal with systems beyond simple ones. Moreover, it will also be interesting to investigate how the number of complex and real equilibria behave for non-generic values of the couplings and frequencies. We plan to study all these issues in future.

In the future, we plan to explore the advantages and disadvantages of both polynomial formulations in terms of the numerical stability, ease to implement, and so on. Systems with sine and cosine of angles and difference of angles are not unique to the Kuramoto model, and hence such a comparative analysis can have far reaching implications. We also point out a remarkable parallel between upper bounds for the equilibria of the Kuramoto model and those for the equilibria of the complete power flow equations [53]–[55]. We plan to quantitatively establish this similarity in the future.

#### APPENDIX

Given  $n$  convex polytopes  $Q_1, \dots, Q_n \subset \mathbb{R}^n$  and positive real numbers  $\lambda_1, \dots, \lambda_n$  Minkowski's Theorem [48] states that the  $n$ -dimensional volume of the *Minkowski sum*  $\lambda_1 Q_1 + \dots + \lambda_n Q_n$ , defined as

$$\{ \lambda_1 q_1 + \dots + \lambda_n q_n \mid q_i \in Q_i \text{ for } i = 1, \dots, n \}$$

is a homogeneous polynomial of degree  $n$  in the variables  $\lambda_1, \dots, \lambda_n$ . The coefficient associated with the monomial  $\lambda_1 \dots \lambda_n$  in this polynomial is known as the *mixed volume* of the polytopes  $Q_1, \dots, Q_n$ .

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